Social Networks

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Motivation

- This project studies the impact of social network on investors' expectation of risk-neutral variance, under a dynamic setup.
- It modifies the model in Han (2018) by adopting the island-connection network in Han and Yang (2013).
- Takeaways: expected volatility of risky asset payoff
  - decreases when investors have larger network
  - increases when more misleading information shared within the socail network

Model Setup

- Two assets economy:
  - Risk-free asset with constant value 1
  - Risky asset with liquidating payoff v at T, paying no dividend
- Noisy supply of risky asset z_t:
  - A random walk with increment $\Delta z_t = z_t - z_{t-\Delta t} \overset{i.i.d}{\sim} N\left(0, \frac{\Delta t}{\rho_z}\right)$
  - Firm has initial supply $\bar{z}$ with shock $\Delta z_0 \sim N\left(0, \frac{\Delta t}{\rho_z}\right)$
- Investor $i \in [0, 1]$:
  - CARA utility with risk-aversion $\gamma > 0$
  - At t, observes an private signal

$$s_{it} = v + \varepsilon_{it}, \text{ with } \varepsilon_{it} \sim N\left(0, \frac{1}{\rho_\varepsilon \Delta t}\right)$$

Social Network

- At t, investor i will be in a group g with other $N-1$ investors, after paying the cost $C(N)$
  - The size of group N could be 1, i.e. the investor didn't join any group. Later, we will show under this setting, the size will be same across different group at t
- Other investors in the same group receive a noisy version of s_{it} trough social communication

$$y_{it}^{g} = s_{it} + \eta_{it}^{g}, \text{ with } \eta_{it}^{g} \sim N\left(0, \frac{1}{\rho_{\eta}\Delta t}\right)$$

- Conditional on v, the precision of y_{it}^{g} from network sharing information is

$$\rho_y \equiv \left(\rho_{\varepsilon}^{-1} + \rho_{\eta}^{-1}\right)^{-1} \in (0, \rho_{\varepsilon})$$

Investor Optimization

- Investor i, with initial wealth W_{i0}, will maximize her expected terminal wealth based on the information available at t

$$\max_{x_{it}} \int -\exp\left\{-\gamma\left(W_t + \sum_{u=t}^{T-\Delta t} x_{iu}(p_{u+\Delta t} - p_u) - \sum_{u=t}^{T-\Delta t} C(N_u)\Delta t\right)\right\}$$
$$\cdot dF\left(v, s_i^t, \ell^t \mid s_{i0}, \ldots, s_{it}, \ell_0, \ldots, \ell_t, y^g_{j\neq i,0}, \ldots, y^g_{j\neq i,t}\right)$$

where the ℓ_t is the public information revealed by the price

Size of Social Network

- **Proposition 1:** The size of of the social network is identical across groups and independent of private or group signals. It satisfies

$$C'(N_t) = \frac{\rho_y}{2\gamma}\mathbb{E}_t^*[\mathrm{Var}_{t+\Delta t}^*(v)]$$

where Var_t^* is the risk-neutral variance at t. C is an increasing convex function of N_t.

Optimal Demand & Pulic Information

- Equilibrium prices p_t is determined by market clearing in the large economy (Schneider (2009))

$$\lim_{G\to\infty} \frac{1}{G} \sum_{g=1}^{G} \left[\frac{1}{N_t} \sum_{i=1}^{N_t} x_{it} \right] = z_t$$

- **Proposition 2 :** Under the equilibrium, the asset demand for each investor i is

$$x_{it} = x_t(\ell_0, \ell_{\Delta t}, ..., \ell_t) + \frac{1}{\gamma} \sum_{u=0}^{t} \left(\rho_\varepsilon s_{iu} + a_u \rho_z \ell_u + \rho_y \sum_{j \neq i}^{N_t} y_{j,y}^{g} \right)$$

  where x_t is a common demand function across i. It is a function of $\{\ell_t\}$ via equilibrium prices $p_t = p_t(\ell_0, \ell_{\Delta t}, ..., \ell_t)$

  - The statistics $\ell_t = a_t v \Delta t - \Delta z_t$ is the public information, where $a_t = \frac{\rho_\varepsilon + (N_t - 1)\rho_y}{\gamma}$ at the equilibrium.
  - Accuracy of public information $\rho_{\ell_t} \frac{[\rho_\varepsilon + (N_t-1)\rho_y]^2}{\gamma^2} \rho_z$

State Variables

- There are 3 state variables that characterizes the system of information content of the market under equilibrium:
  - common component of asset demand x_t
  - expected payoff $m_t \equiv \mathrm{E}\left[y \mid \ell_0, \ell_{\Delta t}, \ldots, \ell_t\right]$
  - total information precision $\tau_t \equiv \sum_{u=0}^{t}(\rho_\varepsilon + a_u^2 \rho_z + (N_u - 1)\rho_y)\Delta t$

SDF

- **Proposition 3 :** A valid SDF is given by the average of normalized marginal utility across investors:

$$\xi_{t,T} = \mathrm{E}\left[\xi^i_{t,T} \mid \ell_0, \ell_{\Delta t}, \ldots, \ell_{T-\Delta t}, v\right] = \mathrm{E}\left[\frac{U'\left(W_{iT}\right)}{\mathrm{E}^i_t\left[U'\left(W_{iT}\right)\right]} \mid \ell_0, \ell_{\Delta t}, \ldots, \ell_{T-\Delta t}, v\right]$$

$$= \exp\left(-\sum_{u=t}^{T-\Delta t}\left[\gamma x_u\left(p_{u+\Delta t} - p_u\right) + \frac{1}{2}\tau_u\left(p^2_{u+\Delta t} - p^2_u\right) - \gamma \cdot C\left(N_u\right)\Delta t\right] - f_t\right)$$

where f_t is a normalizing variable such that $E_t[\xi_{t,T}] = 1$:

$$f_t = \ln \mathrm{E}_t\left[\exp\left(-\sum_{u=t}^{T-\Delta t}\left[\gamma x_u\left(p_{u+\Delta t} - p_u\right) + \frac{1}{2}\tau_u\left(p^2_{u+\Delta t} - p^2_u\right) - \gamma \cdot C\left(N_u\right)\Delta t\right]\right)\right]$$

Continuous Limit

- Using Taylor expansion to $p(t+\Delta t, x_{t+\Delta t}, m_{t+\Delta t}, \tau_{t+\Delta t})$ around (t, x_t, m_t, τ_t), we can have volatility of price p_t in continuous-time limit:

$$\sigma_{pt}^2 = \lim_{\Delta t \to 0} \frac{\mathrm{E}_t\left[(p_{t+\Delta t} - p_t)^2\right]}{\Delta t} = \frac{\partial p}{\partial m}\rho_z^{1/2} a_t h_t - \frac{\partial p}{\partial x}\rho_z^{-1/2}$$

where $h_t = \operatorname{Var}\left[v \mid \ell_0, \ell_{\Delta t}, \ldots, \ell_t\right]$

- Using the same method, we can get the instantaneous drift of p_t

Variance Drift

- The risk-neutral drift of risk-neutral variance $\mathrm{Var}_t^*(v)$ is:

$$\mu_{vt}^* = \lim_{\Delta t \to 0} \frac{\mathrm{E}_t^* \left[\mathrm{Var}_{t+\Delta t}^*(v) - \mathrm{Var}_t^*(v)\right]}{\Delta t} = -\sigma_{pt}^2$$

- **Proposition 4 :** It is decreasing as people has larger network:

$$\frac{\partial \mu_{vt}^*}{\partial N_t} < 0$$

It is increasing as investors share more misleading information within the social network:

$$\frac{\partial \mu_{vt}^*}{\partial \rho_y} < 0$$

Extension

- All the investors are informed in current model. Introducing uninformed investors and endogenous information acquisition, as Grossman and Stiglitz (1980).
- The difficulty lies in calculating expected benefit of private signal in dynamic setting.

Reference

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